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PROPAGATING THERMAL WAVES IN FORCE-COOLED SUPERCONDUCTING DEVICES

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Abstract—The analytical approach of Greene and Saibel [3] to the problem of propagating thermal disturbances in force-cooled superconductors is extended to include all possible wave shapes. The prediction of these moving temperature boundaries is a necessary pre-requisite in understanding the stability characteristics of superconducting devices.

NOMENCLATURE

A_1, cross-sectional area of conductor [cm^2];
A_2, cross-sectional are of helium flow path [cm^2];
A, B, C, constants defined by equation (9);
c_1, specific heat of conductor material [J/gK];
c_2, specific heat of helium at constant pressure [J/gK];
d, length of propagating normal zone downstream from origin of co-ordinates [cm];
D, differential operator;
$G_1 G_2, \ldots$, constants of integration;
h, local surface heat transfer coefficient between conductor and helium coolant [W/cm^2 K];
I^2R, joule heat generated in conductor per unit volume [W/cm^3];
k, thermal conductivity of conductor material [W/cm K];
L, length of propagating normal zone [cm];
S, wetted perimeter of flow passage [cm];
u, length of propagating normal zone upstream from origin of co-ordinates [cm];
v_1, propagating velocity of thermal wave [cm/s];
v_2, helium velocity [cm/s];
x, y, z, cartesian co-ordinates;
$\alpha_1 \alpha_2$, constant exponents defined by equation (11) [cm^{-1}];
Δ, value of θ_1 at $\xi = +\infty$ [K];
θ_1, temperature rise of conductor above equilibrium state [K];
θ_2, temperature rise of helium coolant above equilibrium state [K];
θ_c, temperature difference between equilibrium thermal state and critical temperature of superconductor [K];
ξ, co-ordinate relative to propagating wave [cm];
ρ_1, density of conductor material [g/cm^3];
ρ_2, helium density [g/cm^3].

Subscripts
u, upstream region;
N, normal region;
d, downstream region.

1. INTRODUCTION

SUPERCONDUCTING devices, which employ hollow electrically stabilized composite conductors cooled by the forced flow of helium, are finding increasing use in various areas of technology. These applications include high field magnets, power transmission cables, field windings for motors and generators, etc.

The design of such devices for the thermally stable condition has been discussed elsewhere [1] where digital computing methods were used to make allowance for variations in helium physical properties with pressure and temperature.

Composite superconductors, which include normal metals such as copper or aluminium to stabilize the current, and filamentary "intrinsically stable" superconductors, are designed to eliminate or reduce electrical and thermal disturbances caused by flux jumping. External disturbances can occur, however, which can upset the thermally stable state discussed in [1].

Keilin *et al.* [2] have discussed the problem of thermal instabilities by considering the stationary case where the thermal disturbance or normal region remains stationary relative to the helium coolant. This is equivalent to the case where $v_1 = 0$ in the analysis which follows.

Greene and Saibel [3] considered the limited region of propagating thermal wave fronts which could support normal regions of finite length using the quasi-stationary approach developed by Rosenthal [4].

The present analysis extends the method of Greene and Saibel to examine the full spectrum of possible propagating thermal wave shapes in a force-cooled superconductor subjected to an arbitrary thermal disturbance.

2. DERIVATION OF THE DIFFERENTIAL EQUATIONS FOR A MOVING HEAT SOURCE

The theory of moving heat sources first developed by Rosenthal [4] will be used to assemble the basic differential equations for a thermal wave propagating in the direction of the helium fluid.

Consider the co-ordinate system of Fig. 1 which represents a point heat source moving with velocity v_1, in the x direction.

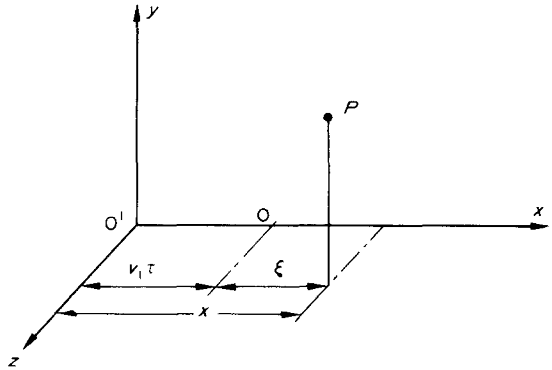


FIG. 1.

After an interval of time τ the origin O′ will move to point O a distance $v_1\tau$ along the x axis.

For uniaxial heat dissipation in the x direction only, the basic heat conduction equation as defined by Jakob [5] is

$$\rho_1 c_1 \frac{\partial t}{\partial \tau} = \frac{\partial}{\partial x}\left(k_x \frac{\partial t}{\partial x}\right) + q''' \qquad (1)$$

where q''' is the internal heat generated per unit volume.

For an observer at point P of Fig. 1 moving with the heat source O we have the quasi-stationary state where

$$\xi = x - v_1 t. \qquad (2)$$

Assuming $k_x = k$ is constant and representing the independent variables in expression (1) in terms of ξ

$$-\rho_1 c_1 v_1 \frac{\partial t}{\partial \xi} = k \frac{\partial^2 t}{\partial \xi^2} + q'''. \qquad (3)$$

Writing expression (3) in terms of the temperature rise θ_1 above the superconductor equilibrium temperature, and putting the internal heat source $q''' = I^2R$ which is the degree of joule heating in the normal state* it follows that

* The condition for normality in a superconductor requires that $\theta_1 \geqslant \theta_c$ where θ_c is the temperature increase required to reach the critical temperature for the superconductor.

$$k\frac{\partial^2\theta_1}{\partial\xi^2} + \rho_1 c_1 v_1 \frac{\partial\theta_1}{\partial\xi} + I^2R = 0. \qquad (4)$$

The solution of equation (4) represents the temperature rise at any section of the superconductor subjected to a moving point heat source I^2R at $\xi = 0$ without losses from the superconductor to the helium coolant. To include the losses to the helium coolant, an additional term which represents the heat exchange from the element $d\xi$ shown in Fig. 2 must be included.

The modified version of equation (4) which

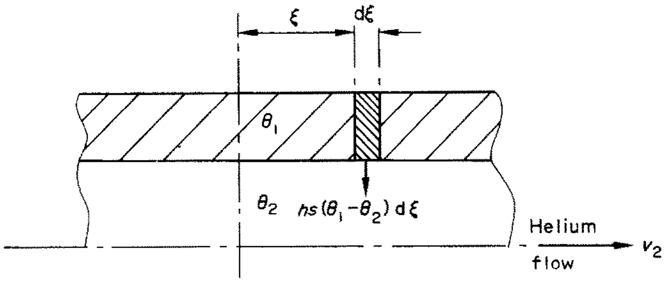


FIG. 2.

includes losses can then be written

$$k\frac{d^2\theta_1}{d\xi^2} + \rho_1 c_1 v_1 \frac{d\theta_1}{d\xi} + I^2R - \frac{hS}{A_1}(\theta_1 - \theta_2) = 0. \qquad (5)$$

To define the energy transfer between superconductor and coolant completely, a second equation in addition to expression (5) is necessary to include the heat gained by the fluid.

Equating the heat gained by a slug of fluid of length $d\xi$ moving with a velocity $v_2 - v_1$ relative to the moving heat source, to the heat lost by the superconductor, gives

$$c_2\rho_2A_2(v_2 - v_1)\frac{d\theta_2}{d\xi}\,.\,d\xi = hS(\theta_1 - \theta_2)\,d\xi$$

i.e.

$$c_2\rho_2A_2(v_2 - v_1)\frac{d\theta_2}{d\xi} - hS(\theta_1 - \theta_2) = 0. \qquad (6)$$

In operator form the differential equations which represent a propagating heat source in the superconductor are therefore

$$\left(kD^2 + \rho_1c_1v_1D - \frac{hS}{A_1}\right)\theta_1 + \frac{hS}{A_1}\theta_2 = -I^2R \qquad (7a)$$

$$[\rho_2c_2A_2(v_2 - v_1)D + hS]\theta_2 - hS\theta_1 = 0. \qquad (7b)$$

The simultaneous linear differential equations (7) are identical with the equations used by Greene and Saibel [3].

3. GENERAL SOLUTION OF THE DIFFERENTIAL EQUATIONS

Multiplying equation (7a) by hS and equation (7b) by $(kD^2 + \rho_1c_1v_1D - hS/A_1)$ to eliminate θ_1, produces θ_2 in the form

$$D(AD^2 + BD + C)\theta_2 = -I^2RhS \qquad (8)$$

where

$$\left.\begin{aligned} A &= \rho_2c_2A_2k(v_2 - v_1)\\ B &= hSk + \rho_1\rho_2c_1c_2A_2v_1(v_2 - v_1)\\ C &= hS\rho_1c_1v_1 - hS\rho_2c_2(A_2/A_1)(v_2 - v_1).\end{aligned}\right\} \qquad (9)$$

The general solution of equation (8) can be found by applying standard methods for linear equations with constant coefficients.

i.e.

$$\theta_2 = G_1 + G_2\,e^{\alpha_1\xi} + G_3\,e^{\alpha_2\xi} - \frac{I^2RhS}{A\alpha_1\alpha_2}\left[\xi + \frac{\alpha_2 + \alpha_1}{\alpha_1\alpha_2}\right] \qquad (10)$$

where

$$\left.\begin{aligned}\alpha_1 &= -\frac{B}{2A} + \frac{1}{2A}[B^2 - 4AC]^{\frac{1}{2}}\\ \alpha_2 &= -\frac{B}{2A} - \frac{1}{2A}[B^2 - 4AC]^{\frac{1}{2}}.\end{aligned}\right\} \quad (11)$$

A, B and C are defined by expressions (9) and G_1, G_2 and G_3 are constants of integration to be found from the boundary conditions.

Substituting (10) in the second of equations (7) the temperature rise in the superconductor is given by

$$\theta_1 = G_1 + \left(1 + \frac{A\alpha_1}{khS}\right) G_2\, e^{\alpha_1 \xi} + \left(1 + \frac{A\alpha_2}{khS}\right) G_3\, e^{\alpha_2 \xi} - \frac{I^2RhS}{A\alpha_1\alpha_2}\left[\xi + \frac{\alpha_1 + \alpha_2}{\alpha_1\alpha_2} + \frac{A}{khS}\right]. \quad (12)$$

Equations (10) and (12) define the temperature rise of the helium fluid and conductor, respectively, as a point heat source propagates throughout the superconductor length.

4. LIMITING CONDITIONS IMPOSED ON THE GENERAL SOLUTION

Before suitable boundary conditions can be applied to find the constants in expressions (10) and (12) the signs for α_1 and α_2 must be known. It is seen from equations (9) and (11), that the values of α_1 and α_2 depend on the magnitudes of the velocities v_1 and v_2.

A closer examination of the values A, B and C shows that five different conditions exist which affect the signs for α_1 and α_2. These five limiting conditions are summarized in Table 1.

Each of the five cases listed in Table 1 can in principle provide solutions for different propagating thermal wave shapes. Each case was examined in detail because of the uncertainties which surround the mechanisms causing a thermal wave to propagate at some velocity v_1.

5. BOUNDARY CONDITIONS AND THE NORMAL REGION

If a flux jump, or some other source of electrothermal disturbance, occurs at an arbitrary point $\xi = 0$ along the superconductor length causing it to go "normal" at that point, then joule heating of magnitude I^2R occurs due to the finite resistance R

After some interval of time the normal region can expand, under certain special conditions to be discussed later, and the temperature rise in the superconductor will develop into a profile of the form shown in Fig. 3.

This profile was discussed in [2] for the stationary condition $v_1 = 0$ and examined in

Table 1.

Case no.	Limiting condition	A	B	C	α_1	α_2	Remarks
1	$v_2 = v_1$	0	+	+	∞	∞	D.E. (8) is $D(BD + C)\theta_2 = -I^2RhS$
2	$v_2 > \left(1 + \frac{\rho_1 C_1 A_1}{\rho_2 C_2 A_2}\right) v_1$	+	+	−	+	−	$\lvert\alpha_2\rvert > \lvert\alpha_1\rvert$
3	$v_2 = \left(1 + \frac{\rho_1 C_1 A_1}{\rho_2 C_2 A_2}\right) v_1$	+	+	0	0	$-\frac{B}{A}$	D.E. is $D^2(AD + B)\theta_2 = -I^2RhS$
			0				$\alpha_1 = -\sqrt{(C/A)}$ $\alpha_2 = +\sqrt{(C/A)}$
4	$v_2 < v_1$	−	− +	+	−	+	
5	$v_1 < v_2 < \left(1 + \frac{\rho_1 C_1 A_1}{\rho_2 C_2 A_2}\right) v_1$	+	+	+	$-B/2A$ − i	$-B/2A$ − i	$B^2 = 4AC$ $B^2 > 4AC$ $B^2 < 4AC$

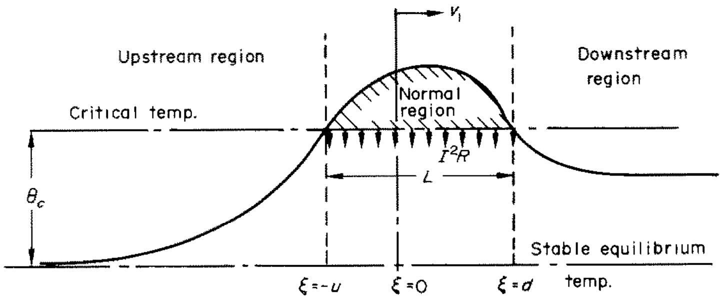


FIG. 3.

detail by Greene and Saibel [3] for the propagating case.

Whereas Keilin *et al.* [2] indicate in their Fig. 2b that the normal region is a discontinuous function, Greene and Saibel assume a continuous function for the temperature distribution throughout the normal region. Although the origin of the thermal disturbance is located at some point, the nature of the energy spread would suggest that a continuous function throughout the normal region is the correct approach.

The length of the normal region $L = u + d$ requires that $\theta_1 > \theta_c$ where the critical temperature rise θ_c is a function of the magnetic field strength in the superconductor region.

At sections upstream and downstream of the normal region where $\theta_1 < \theta_c$ the conductor remains superconducting and the I^2R terms in equations (10) and (12) are consequently zero.

The constants of integration G_1, G_2 and G_3 appearing in equations (10) and (12) will be different for each of the three regions shown in Fig. 3 making a total of nine constants and therefore nine required boundary conditions.

Eight of the required boundary conditions are straightforward describing the conditions at $\xi = \pm\infty$ and the matching of the temperatures, temperature gradients and the jump in heat flow at $\xi = -u$ and $\xi = +d$.

These eight conditions are

$$\text{at} \quad \xi = \pm\infty, \quad \theta_1 = 0 \quad \text{or} \quad \frac{d\theta_1}{d\xi} = 0$$

depending on which of the five cases of Table 1 is being considered.

$$\begin{aligned} \text{at} \quad \xi = d, \quad & (\theta_1)_d = (\theta_1)_N \\ & \left(\frac{d\theta_1}{d\xi}\right)_d = \left(\frac{d\theta_1}{d\xi}\right)_N \\ & k\left(\frac{d^2\theta_1}{d\xi^2}\right)_d = k\left(\frac{d^2\theta_1}{d\xi^2}\right)_N + I^2R \\ \text{at} \quad \xi = -u, \quad & (\theta_1)_u = (\theta_1)_N \\ & \left(\frac{d\theta_1}{d\xi}\right)_u = \left(\frac{d\theta_1}{d\xi}\right)_N \\ & k\left(\frac{d^2\theta_1}{d\xi^2}\right)_u = k\left(\frac{d^2\theta_1}{d\xi^2}\right)_N + I^2R. \end{aligned} \tag{13}$$

The remaining boundary condition necessary to completely define the problem is less straightforward and is only required when discussing the solution for case No. 2 of Table 1 and it is due to Greene and Saibel [3]. This condition applies the conservation of energy principle to the complete system to find the value of L as a function of the asymptotic value Δ (i.e. $\Delta = -I^2RhSL/C$).

6. PROPAGATING WAVE SHAPES

In applying the boundary conditions (13), the constants of integration for the three conductor regions shown in Fig. 3 will be identified as follows

G_1' G_2' G_3' upstream region

G_1'' G_2'' G_3'' normal region

$G_1'''G_2'''G_3'''$ downstream region

Case 1. $v_1 = v_2$

The general solution for the temperature profiles in the coolant and conductor in this case is

$$\theta_1 = \theta_2 = G_1 + G_2\,e^{-(C/B)\xi} + \frac{I^2RhS}{C}\left(\frac{B}{C} - \xi\right). \qquad (14)$$

Applying the boundary conditions (13) for the three regions to evaluate the constants $G_1'G_2'G_1''\ldots$ gives the complete solution

$$(\theta_1)_u = \frac{I^2RhS}{C}.d + \frac{A_1I^2R(B/C - \xi)}{v_1(\rho_1c_1A_1 + \rho_2c_2A_2)}$$

$$(\theta_1)_N = \frac{I^2RhS}{C}\left(d + \frac{B}{C} - \xi\right) + \frac{B}{C}\left[\frac{A_1I^2R}{v_1(\rho_1c_1A_1 + \rho_2c_2A_2)} - \frac{I^2RhS}{C}\right] \times e^{-(C/B)\xi} \qquad (15)$$

$$(\theta_1)_d = \left[\frac{A_1I^2R}{v_1(\rho_1c_1A_1 + \rho_2c_2A_2)} + \frac{I^2RhS}{C}(e^{(C/B)d} - 1)\right]\frac{B}{C}.\,e^{-(C/B)\xi}$$

where the length of penetration of the normal region d, downstream from $\xi = 0$ is given by

$$d = -\frac{B}{C}\ln\frac{\left[\dfrac{C\theta_c}{B} - \dfrac{I^2RhS}{C}\right]}{\left[\dfrac{A_1I^2R}{v_1(\rho_1c_1A_1 + \rho_2c_2A_2)} - \dfrac{I^2RhS}{C}\right]}. \qquad (16)$$

The curves of Fig. 4 illustrate the results of applying the foregoing equations to the numerical case of a superconductor proposed for use in a bubble chamber high field magnet. The temperature profile of the moving thermal front has been calculated for three different propagating velocities $v_1 = v_2$.

The limiting cases for the shape of the moving wave front when $v_2 = v_1$ can be ascertained from equation (16).

$$d = 0 \quad \text{when} \quad \frac{C\theta_c}{B} = \frac{A_1I^2R}{v_1(\rho_1c_1A_1 + \rho_2c_2A_2)}$$

or using $C = hS\rho_1c_1v_1$, $B = hSk$

$d = 0$ when

$$v_1 = \sqrt{\left(\frac{kA_1I^2R}{\theta_c\rho_1c_1(\rho_1c_1A_1 + \rho_2c_2A_2)}\right)}. \qquad (17)$$

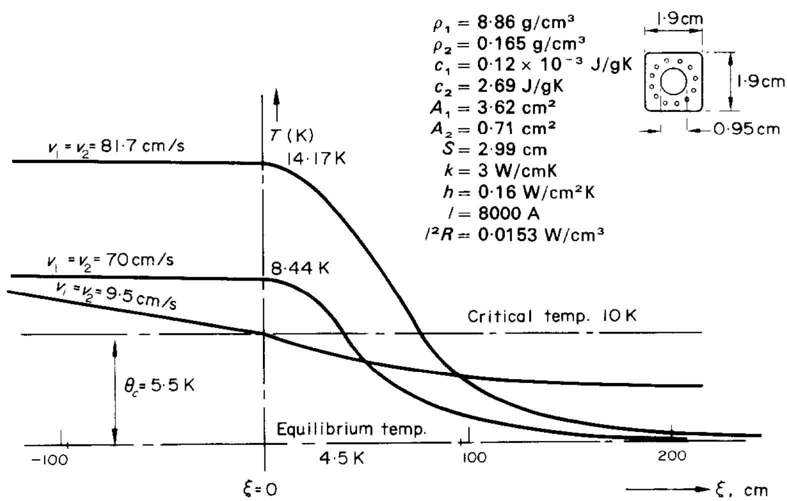


FIG. 4. Typical temperature profiles for case 1. Calculated for a superconducting winding for use in a bubble chamber high field magnet.

For the particular case illustrated in Fig. 5 equation (17) gives a value of $v_1 = v_2 = 9{\cdot}5$ cm/s. At this velocity there will be no normal region preceding the moving wave front but an almost linear temperature gradient along the length of the coil.

Again by inspection of equation (16)

$$d \to \infty \quad \text{as} \quad \frac{C\theta_c}{B} \to \frac{I^2RhS}{C}$$

which gives

$$v_1 = \frac{1}{\rho_1 c_1}\sqrt{\left(\frac{kI^2R}{\theta_c}\right)}. \tag{18}$$

For the data of Fig. 4, equation (18) gives a value of $v_1 = 85{\cdot}9$ cm/s at which velocity the complete superconductor goes normal. Equation (18) is identical with the expression given by Greene and Saibel [3] for $v_D = v_2$ for maximum propagation of the downstream edge of the normal region.

Case 2. $v_2 > (1 + \rho_1 c_1 A_1/\rho_2 c_2 A_2)v_1$

The solution for this case, where a finite propagating normal region can be sustained along the superconductor, is due to Greene and Saibel [3].

Greene and Saibel's solution is

$$(\theta_1)_u = \theta_c \,\mathrm{e}^{\alpha_1\xi}$$

$$(\theta_1)_N = G_1'' + G_2''\,\mathrm{e}^{\alpha_1\xi} + G_3''\,\mathrm{e}^{\alpha_2\xi} + G_4''\xi \tag{19}$$

$$(\theta_1)_d = \Delta + (\theta_c - \Delta)\,\mathrm{e}^{\alpha_2(\xi - L)}.$$

The constant G_4 appears in the above solution by writing equation (12) in its most general form.

$$(\theta_2)_u = \frac{\theta_c\,\mathrm{e}^{\alpha_1\xi}}{(1 + A\alpha_1/khS)}$$

$$\begin{aligned}(\theta_2)_N = {} & G_1''(1 - \mathrm{e}^{-(khS/A)\xi}) \\ & + \frac{G_2''}{(1 + A\alpha_1/khS)}(\mathrm{e}^{\alpha_1\xi} - \mathrm{e}^{-(khS/A)\xi}) \\ & + \frac{G_3''}{(1 + A\alpha_2/khS)}(\mathrm{e}^{\alpha_2\xi} - \mathrm{e}^{-(khS/A)\xi}) \\ & + G_4''\left(\xi - \frac{A}{khS} + \frac{A\,\mathrm{e}^{-(khS/A)\xi}}{khS}\right) \\ & + \frac{\theta_c\,\mathrm{e}^{-(khS/A)\xi}}{(1 + A\alpha_1/khS)}\end{aligned} \tag{20}$$

$$\begin{aligned}(\theta_2)_d = {} & \Delta + [(\theta_2)_{N(\xi = L)} - \Delta]\,\mathrm{e}^{(khS/A)(L - \xi)} \\ & + \frac{\theta_c - \Delta}{(1 + A\alpha_2/khS)} \times [\mathrm{e}^{\alpha_2(\xi - L)} - \mathrm{e}^{(khS/A)(L - \xi)}]\end{aligned}$$

where

$$\begin{aligned} G_1'' &= \frac{\theta_c - \Delta}{1 - \mathrm{e}^{\alpha_2 L}} + \frac{\theta_c}{1 - \mathrm{e}^{-\alpha_1 L}} \\ G_2'' &= -\frac{\theta_c\,\mathrm{e}^{-\alpha_1 L}}{1 - \mathrm{e}^{-\alpha_1 L}} \\ G_3'' &= -\frac{\theta_c - \Delta}{1 - \mathrm{e}^{\alpha_2 L}} \\ G_4'' &= \frac{\Delta}{L}\end{aligned} \tag{21}$$

and the asymptotic value of θ_1 at $\xi = +\infty$ is

$$\Delta = \frac{\theta_c\left[\dfrac{\alpha_1 L}{1 - \mathrm{e}^{-\alpha_1 L}} + \dfrac{\alpha_2 L}{1 - \mathrm{e}^{\alpha_2 L}}\right]}{\left[1 + \dfrac{\alpha_2 L}{1 - \mathrm{e}^{\alpha_2 L}}\right]}. \tag{22}$$

The length of the normal propagating region L is found from the expression

$$\frac{\alpha_1 L}{1 - \mathrm{e}^{-\alpha_1 L}}\left[\frac{1 + \dfrac{\alpha_2 L}{1 - \mathrm{e}^{\alpha_2 L}}}{\dfrac{\alpha_1 L}{1 - \mathrm{e}^{-\alpha_1 L}} + \dfrac{\alpha_2 L}{1 - \mathrm{e}^{\alpha_2 L}}}\right] = -\frac{\alpha_2}{\alpha_1 - \alpha_2}\left[1 + \frac{\rho_2 c_2 A_2(v_2 - v_1)\alpha_1}{hS}\right]. \tag{23}$$

Limitations of the value of L

As shown in Table 1 for case 2, α_1 is positive and α_2 negative and consequently the right hand side of expression (23) is always positive. Solutions to equation (23) which yield $L > 0$ will therefore only exist if the left hand side of this equation is greater than zero.

Expanding the exponential terms on the LHS of equation (23) in power series shows that

$$\frac{\alpha_1 L}{1 - e^{-\alpha_1 L}}\left[\frac{1 + \dfrac{\alpha_2 L}{1 - e^{\alpha_2 L}}}{\dfrac{\alpha_1 L}{1 - e^{-\alpha_1 L}} + \dfrac{\alpha_2 L}{1 - e^{\alpha_2 L}}}\right] \to \frac{\alpha_2}{\alpha_2 + \alpha_1}$$

as $L \to 0$.

Solutions of expression (23) which produce positive values for L therefore require that

$$-\frac{\alpha_2}{\alpha_1 - \alpha_2}\left[1 + \frac{\rho_2 c_2 A_2 (v_2 - v_1)\alpha_1}{hS}\right] > \frac{\alpha_2}{\alpha_1 + \alpha_2}. \quad (24)$$

Using the values for α_1 and α_2 given by equations (11) the condition (24) simplifies to

$$v_1(v_2 - v_1) > \frac{khS}{\rho_1 c_1 \rho_2 c_2 A_2}. \quad (25)$$

Condition (25) is a very important aspect of the Greene and Saibel [3] solution and provides the limitations for case 2 beyond which a finite propagating normal region cannot be sustained.

Using the equality in equation (25), the upper and lower limits for finite L are

$$v_1 = \frac{v_2}{2} \pm \sqrt{\left[\left(\frac{v_2}{2}\right)^2 - \frac{khS}{\rho_1 c_1 \rho_2 c_2 A_2}\right]} \quad (26)$$

and for real solutions to exist the term under the square root in equation (26) must be positive, i.e.

$$v_2 > 2\sqrt{\left(\frac{khS}{\rho_1 c_1 \rho_2 c_2 A_2}\right)}. \quad (27)$$

Outside the limits of expression (26) valid solutions exist for the case where

$$v_2 > \left(1 + \frac{\rho_1 c_1 A_1}{\rho_2 c_2 A_2}\right) v_1.$$

Indeed such solutions must exist since propagation along the super conductor at a velocity v_1 above or below the limits of equation (26) must precede the appearance of a finite normal region.

For propagation without a finite normal region the upstream and downstream temperature distributions for this case are

$$(\theta_1)_u = \theta_c e^{\alpha_1 \xi}$$

$$(\theta_1)_d = \theta_c + \frac{1}{\alpha_2}\left(\alpha_1 \theta_c - \frac{I^2 R}{k}\right)(e^{\alpha_2 \xi} - 1) \quad (28)$$

$$(\theta_2)_u = \frac{\theta_c e^{\alpha_1 \xi}}{(1 + A\alpha_1/khS)}$$

$$(\theta_2)_d = \theta_c + \frac{1}{\alpha_2}\left(\alpha_1 \theta_c - \frac{I^2 R}{k}\right)$$

$$\times \left[\frac{e^{\alpha_2 \xi}}{1 + \dfrac{A\alpha_2}{khS}} + \frac{e^{-(khS/A)\xi}}{1 + \dfrac{khS}{A\alpha_2}} - 1\right]$$

$$- \frac{\theta_c e^{-(khS/A)\xi}}{\left(1 + \dfrac{khS}{A\alpha_1}\right)}. \quad (29)$$

Because α_2 is negative in this case, the second of equations (28) will give a value for $(\theta_1)_d > \theta_c$ if the term $\alpha_1 \theta_c - I^2 R/k$ is positive. This would violate the condition of no finite region of normality and consequently expressions (28) and (29) are only valid provided

$$\alpha_1 \leqslant \frac{I^2 R}{k\theta_c}. \quad (30)$$

If the coolant velocity exceeds the velocity of propagation of the thermal wave by an amount which makes $\alpha_1 > I^2 R/k\theta_c$ then the temperature excess region will disappear and the system will be fully stable. Figure 5 summarizes the results for the special case where $v_2 > (1 + \rho_1 c\, A_1/\rho_2 c\, A_2) v_1$ in the limited regions

$$0 \leqslant v_1 \leqslant \frac{v_2}{2} - \sqrt{\left[\left(\frac{v_2}{2}\right)^2 - \frac{khS}{\rho_1 c_1 \rho_2 c_2 A_2}\right]}$$

and

$$v_1 \geqslant \frac{v_2}{2} + \sqrt{\left[\left(\frac{v_2}{2}\right)^2 - \frac{khS}{\rho_1 c_1 \rho_2 c_2 A_2}\right]}.$$

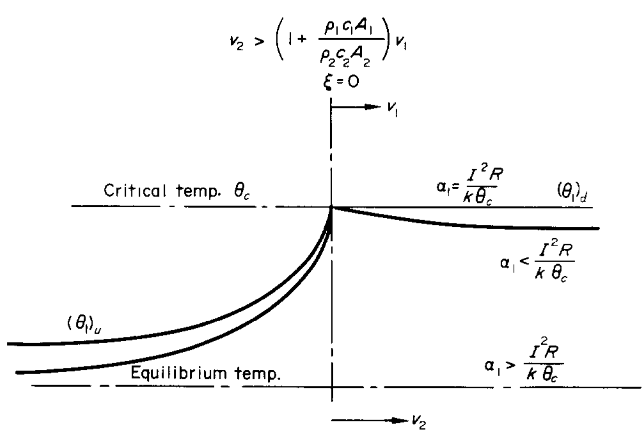


FIG. 5. Propagating point source for case 2.

As v_1 increases α_1 reduces in value and consequently the requirement of equation (30) will generally be satisfied by the latter of the above two regions.

Variations of the propagating normal zone length for case 2 are illustrated in Fig. 6.

Case 3. $v_2 = (1 + \rho_1 c_1 A_1/\rho_2 c_2 A_2) v_1$

Applying the conservation of energy principle throughout the superconductor length from $\xi = -\infty$ to $\xi = +\infty$ shows that no finite propagating length can exist in this case.

The complete solution for the propagating

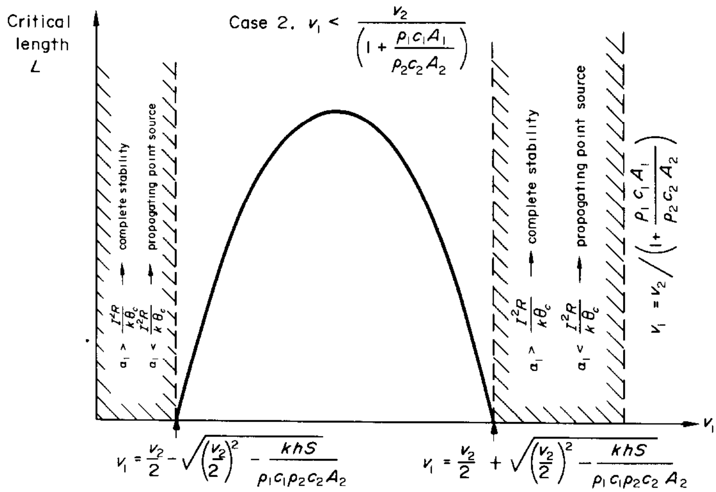


FIG. 6. Variations in propagating normal region length for case 2.

point source in this case is

$$(\theta_1)_u = \theta_c$$

$$(\theta_1)_d = \theta_c - \frac{AI^2R}{B}(1 - e^{-(B/A)\xi}) \quad (31)$$

$$(\theta_2)_u = \theta_c + \frac{AI^2R}{k(khS - B)}$$

$$(\theta_2)_d = \theta_c - \frac{AI^2R}{kB}\left[1 - \frac{e^{-(B/A)\xi}}{\dfrac{B}{khS}}\right]. \quad (32)$$

*Case.*4 $v_1 > v_2$

The complete solution for this case is

$$(\theta_1)_u = \theta_c - \frac{1}{k\alpha_2}(I^2R + k\alpha_1\theta_c)(1 - e^{\alpha_2\xi})$$

$$(\theta_1)_d = \theta_c e^{\alpha_1\xi} \quad (33)$$

$$(\theta_2)_u = \frac{\theta_c}{(1 + A\alpha_1/khS)} - \frac{1}{k\alpha_2}(I^2R + k\alpha_1\theta_c) \times \left(\frac{1 - e^{\alpha_2\xi}}{1 + A\alpha_2/khS}\right)$$

$$(\theta_2)_d = \frac{\theta_c e^{\alpha_1\xi}}{(1 + A\alpha_1/khS)}. \quad (34)$$

Since α_1 is negative in this case, expressions (33) and (34) are only valid provided

$$|\alpha_1| \leqslant \frac{I^2R}{k\theta_c}. \quad (35)$$

Case 5.$v_1 < v_2 < (1 + \rho_1 c_1 A_1/\rho_2 c_2 A_2)v_1$

As shown by Table 1 there are three possible solutions in this case depending on the relative values for B^2 and $4AC$.

It can be shown, however, that in this case a propagating point source can only be sustained provided $B^2 = 4AC$.

The following solution is obtained in this case

$$(\theta_1)_u = \theta_c$$

$$(\theta_1)_d = \theta_c - \frac{2AI^2R}{kB}(1 - e^{-(B/2A)\xi}) \quad (36)$$

and the corresponding rise in helium coolant temperature is

$$(\theta_2)_u = \theta_c + \frac{AI^2R}{k(khS - B/2)}$$

$$(\theta_2)_d = \theta_c - \frac{2AI^2R}{kB}\left[1 - \frac{e^{-(B/2A)\xi}}{1 + B/2khS}\right]. \quad (37)$$

Expressions (37) are only valid provided $B/2 > khS$ otherwise the coolant temperature on the upstream side of the moving point disturbance would be greater than the conductor temperature.

Using the value of B from the second of expressions (9) the same limiting condition expressed by equation (25) is obtained.

Rearranging the limitation (25)

$$v_2 > v_1 + \frac{khS}{\rho_1 c_1 \rho_2 c_2 A_2 v_1}.$$

Whereas the above requirement for v_2 is always satisfied in case 2 it is not always satisfied in the present case where

$$v_2 < v_1 + \frac{\rho_1 c_1 A_1}{\rho_1 c_2 A_2} . v_1 .$$

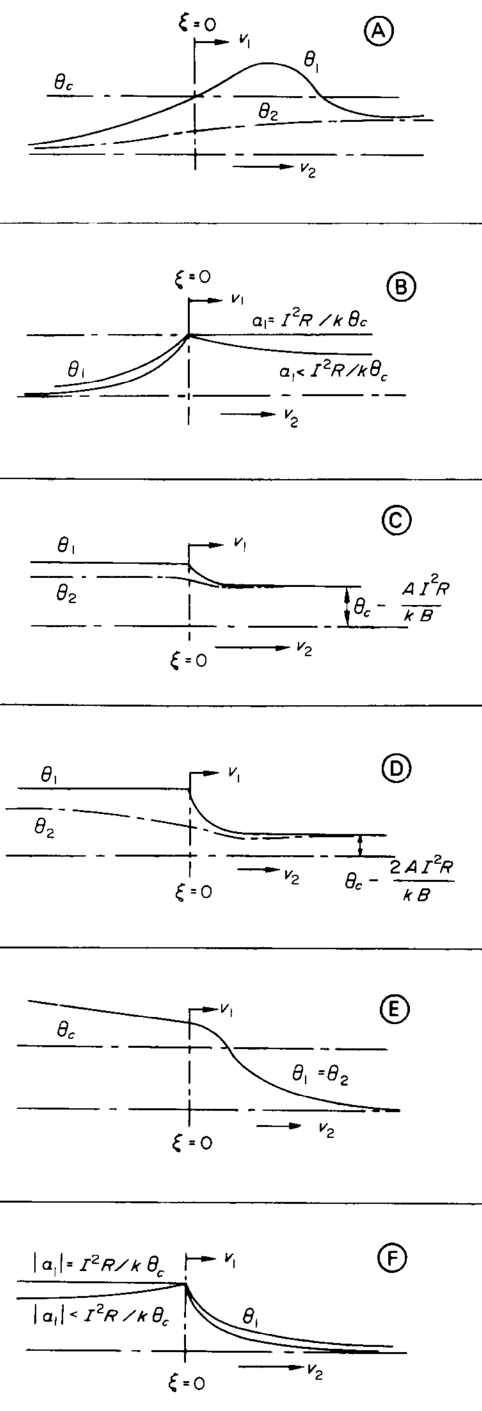
The validity of equations (36) and (37) therefore require the further constraint

$$v_1 > \frac{1}{\rho_1 c_1 A_1}\sqrt{(khS)}. \quad (38)$$

Summary of propagating thermal wave shapes

Figure 7 summarizes the theoretically possible thermal wave shapes derived in the foregoing sections. These profiles have been drawn in ascending order of propagating velocity which is equivalent to travelling from left to right along the axis of Fig. 8.

From a practical point of view it appears from Fig. 8 that only in cases 1 and 2 can finite regions of normality propagate along the conductor. The other propagating wave shapes require the continued existence of a point heat source which is inherently unstable and would normally disappear.



Ⓐ Case 2 $v_2 > \left(1 + \frac{\rho_1 c_1 A_2}{\rho_2 c_2 A_2}\right) v_1$

$$v_1 > \frac{v_2}{2} - \sqrt{\left(\frac{v_2}{2}\right)^2 - \frac{khs}{\rho_1 c_1 \rho_2 c_2 A_2}}$$

or $$v_1 < \frac{v_2}{2} + \sqrt{\left(\frac{v_2}{2}\right)^2 - \frac{khs}{\rho_1 c_1 \rho_2 c_2 A_2}}$$

Ⓑ Case 2. $v_2 > \left(1 + \frac{\rho_1 c_1 A_1}{\rho_2 c_2 A_2}\right) v_1$

$$0 \leqslant v_1 \leqslant \frac{v_2}{2} - \sqrt{\left(\frac{v_2}{2}\right)^2 - \frac{khs}{\rho_1 c_1 \rho_2 c_2 A_2}}$$

$$v_1 \geqslant \frac{v_2}{2} + \sqrt{\left(\frac{v_2}{2}\right)^2 - \frac{khs}{\rho_1 c_1 \rho_2 c_2 A_2}}$$

$$\alpha_1 \leqslant I^2R/k\theta_c$$

Ⓒ Case 3. $v_2 = \left(1 + \frac{\rho_1 c_1 A_1}{\rho_2 c_2 A_2}\right) v_1$

Ⓓ Case 5. $v_1 < v_2 < \left(1 + \frac{\rho_1 c_1 A_1}{\rho_2 c_2 A_2}\right) v_1$

$$B^2 = 4AC$$

$$v_1 > \frac{\sqrt{(khs)}}{\rho_1 c_1 A_1}$$

Ⓔ Case 1. $v_1 = v_2$

Ⓕ Case 4. $v_1 > v_2$

$$|\alpha_1| \leqslant I^2R/k\theta_c$$

FIG. 7. Summary of propagating thermal wave shapes.

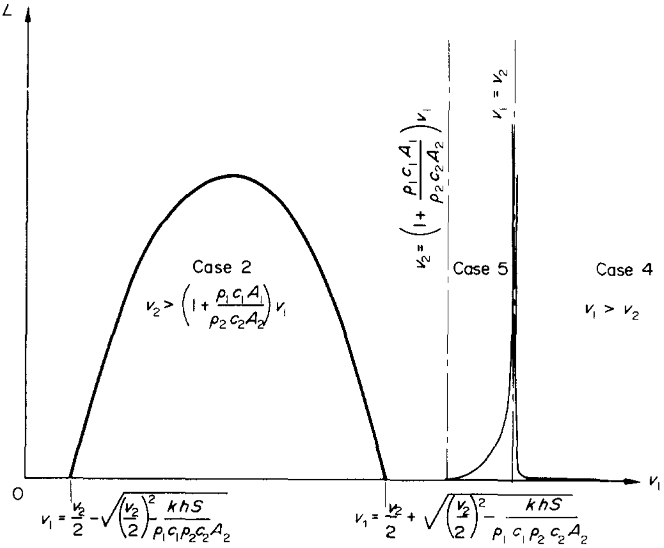


FIG. 8. Summary of variations in propagating normal region length.

However, to travel from the condition of case 2 to case 1, or vice versa, the profiles pertaining to the point source of heat will probably exist. In other words these source conditions will play a large part in determining whether or not significant thermal instabilities take place.

Referring again to Fig. 7, initiation of a thermal wave front having the profile Ⓑ can occur as the propagating velocity increases from rest to a value of

$$v_1 = \frac{v_2}{2} - \sqrt{\left[\left(\frac{v_2}{2}\right)^2 - \frac{khs}{\rho_1 c_1 \rho_2 c_2 A_2}\right]}.$$

At this value a propagating normal zone starts to develop into the profile Ⓐ which initially increases in length and then contracts back to profile Ⓑ at velocity

$$v_1 = \frac{v_2}{2} + \sqrt{\left[\left(\frac{v_2}{2}\right)^2 - \frac{khS}{\rho_1 c_1 \rho_1 c_2 A_2}\right]}.$$

As the propagating velocity v_1 continues to increase the thermal profile changes to Ⓒ at $v_1 = v_2/(1 + \rho_1 c_1 A_1/\rho_2 c_2 A_2)$ and then to profile Ⓓ beyond this value.

When the propagating velocity reaches the same velocity as the helium coolant a continuing trail of normality Ⓔ will develop if this velocity is maintained. If, however, the thermal wave is accelerated through this condition, a point source having the profile Ⓕ will be established which disappears when $\alpha_1 > I^2R/k\theta_c$.

7. CONCLUSIONS

Propagating point sources of "normal" energy can theoretically exist outside the limits of the finite propagating normal regions defined by Greene and Saibel. The initial velocity with which a thermal disturbance originates within a superconducting device is unknown at the present time. If, however, this disturbance starts from rest, or at some initial velocity outside the limits of finite normal length propagation, then the thermal wave shapes summarised in Fig. 7 will exist during the transitionary periods.

In the cases where a constant temperature rise θ_c is predicted upstream or downstream of the propagating point source, the solutions require that the conductor remains superconducting everywhere except at the point source $\xi = 0$. Obviously any slight thermal disturbance superimposed on the moving thermal wave could upset the quasi-stable wave in either direction.

Computer programmes exist [6] which calculate the propagating thermal wave shapes discussed in this paper taking into account the variable properties of the helium coolant. Initial runs with these programmes show that finite propagating regions can appear owing to changes in helium density with a corresponding change in the coolant velocity.

Experimental measurements on superconductors with moving heat sources are necessary to extend the experimental work of Keilin *et al.* [2], and verify, or otherwise, the propagating thermal waves predicted by theory.

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PROPAGATION D'ONDES THERMIQUES DANS DES DISPOSITIFS SUPRACONDUCTEURS

Résumé—L'approche analytique de Green et Saibel pour le problème de perturbations thermiques en propagation dans les supraconducteurs est étendue à toutes les formes possibles d'onde. L'estimation de ces limites thermiques mobiles est nécessaire à la compréhension des caractéristiques de stabilité des dispositifs supraconducteurs.

FORTSCHREITENDE WÄRMEWELLEN IN ERZWUNGEN GEKÜHLTEN SUPRALEITENDEN ANORDNUNGEN

Zusammenfassung—Die analytische Näherung von Greene und Saibel [3] zum Problem der fortschreitenden thermischen Störungen in erzwungen gekühlten Supraleitern wird so ausgeweitet, dass sie alle möglichen Wellenformen einschliesst. Die Voraussage über diese wandernden Temperaturgrenzen ist eine notwendige Voraussetzung zum Verstehen der Stabilitätskriterien von supraleitenden Einrichtungen.

РАСПРОСТРАНЕНИЕ ТЕПЛОВЫХ ВОЛН В СВЕРХПРОВОДЯЩИХ УСТРОЙСТВАХ С ВЫНУЖДЕННЫМ ОХЛАЖДЕНИЕМ

Аннотация—Аналитический метод Грина и Сэйбела [3] для задачи распространения тепловых возмущений в сверхпроводниках с вынужденным охлаждением обобщается на случай всевозможных форм волны. Определение подвицных температурных границ является необходимой предпосылкой для понимания характ ристик устойчивости сверхпроводящих устройств.